Linear Lower Bounds for $\delta_c(p)$ for a Class of $2D$ Self-Destructive Percolation Models

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Abstract

The self-destructive percolation model is defined as follows: Consider percolation with parameter $p > p_c$. Remove the infinite occupied cluster. Finally, give each vertex (or, for bond percolation, each edge) that at this stage is vacant, an extra chance δ to become occupied. Let $\delta_c(p)$ be the minimal value of δ, needed to obtain an infinite occupied cluster in the final configuration. This model was introduced some years ago by van den Berg and Brouwer. They showed that, for the site model on the square lattice (and a few other $2D$ lattices satisfying a special technical condition) that $\delta_c(p) \geq \frac{(p-p_c)}{p}$. In particular, $\delta_c(p)$ is at least linear in $p - p_c$.

Although the arguments used by van den Berg and Brouwer look quite rigid, we show that they can be suitably modified to obtain similar linear lower bounds for $\delta_c(p)$ (with p near p_c) for a much larger class of $2D$ lattices, including bond percolation on the square and triangular lattices, and site percolation on the star lattice (or matching lattice) of the square lattice.

Keywords: percolation; self-destructive percolation; critical value

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1 Introduction

Some years ago van den Berg and Brouwer, motivated by the study of forest-fire processes, introduced the self-destructive percolation model (see [1] and [3]). This model can be described in a few steps as follows.

Let $G = (\mathbb{V}, \mathbb{E})$ be a connected, infinite, locally finite graph. The first step is to perform independent site percolation with parameter p on this graph: we declare each vertex occupied with probability p and vacant with probability $1-p$, independent of the other vertices. If $U, W \subset \mathbb{V}$, we use the notation $\{U \leftrightarrow W\}$ for the event that there is a path of occupied vertices connecting some vertex of U to some vertex of W, and $\{U \leftrightarrow \infty\}$ for the event that there exists an infinite path of occupied vertices starting from some vertex in U. If $G = (\mathbb{V}, \mathbb{E})$ is transitive (i.e., without lost of generality we can choose any vertex to be the origin), let $\theta(p)$ be the probability that the origin belongs to an infinite occupied cluster.

Since $\theta(p)$ is monotone in p, there is a critical value $p_c \in [0, 1]$ such that $\theta(p) > 0$ if $p \in (p_c, 1]$ and $\theta(p) = 0$ if $p \in [0, p_c)$. It is well-known (see [4] for this and other background results on percolation) that $\theta(p_c) = 0$ for a large class of graphs including the d-dimensional hypercubic lattice, for $d = 2$ or $d \geq 19$, the triangular and hexagonal lattices.

Now suppose that, by some 'catastrophe', the infinite occupied clusters are destroyed, that is, each vertex that belongs to an infinite occupied cluster becomes vacant. After the catastrophe we give each vacant vertex an extra chance to become occupied; more precisely, each vacant vertex becomes occupied with probability δ, independent of everything else. Let $P_{p,\delta}$ be the distribution of the final configuration and $\theta(p, \delta) := P_{p,\delta}(0 \leftrightarrow \infty)$.

An equivalent but more formal (and often more convenient) description of the self-destructive percolation model is the following: let $X_i,\ i \in \mathbb{V}$ be a sequence of i.i.d. 0-1 random variables with parameter p, and let $Y_i,\ i \in \mathbb{V}$ be another sequence of i.i.d. 0-1 random variables with parameter δ. (Here, we can interpret 0 as vacant and 1 as occupied). Moreover we take the sequence of Y_i's independent of the sequence of X_j's. Let $X_i^*,\ i \in \mathbb{V}$ be defined by

$$X_i^* = \begin{cases} 1, & \text{if } X_i = 1 \text{ and there is no infinite } X\text{-occupied path from } i \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

Here, by X-occupied path we mean a path on which each vertex j has $X_j = 1$. Finally, we define $Z_i := X_i^* \vee Y_i$. This sequence $Z_i,\ i \in \mathbb{V}$ is the final

configuration and the measure $P_{p,\delta}$ is its distribution. Analogously (with obvious modifications of the definitions) one can define the self-destructive bond percolation model.

Monotonicity in δ obviously implies that for each $p \in [p_c, 1]$ there exists a $\delta_c(p) \in [0,1]$ such that $\theta(p,\delta) = 0$ for all $\delta \in [0, \delta_c(p))$ and $\theta(p,\delta) > 0$ for all $\delta \in (\delta_c(p), 1]$. It is also clear that if $\delta > p_c$, then $\theta(p,\delta) > 0$ for all $p \in [0,1]$; that is,

$$\forall p \ \delta_c(p) \leq p_c. \tag{1.2}$$

As to lower bounds for p_c, Proposition 3.1 of [1] and Proposition 2.3.2 in [3] show that for site percolation on the square (and the triangular and the honeycomb) lattice $\delta_c(p)$ is at least linear in $p - p_c$. More precisely it was shown for these models that

$$\text{If } p(1-\delta) > p_c, \text{ then } \theta(p,\delta) = 0, \tag{1.3}$$

which is obviously equivalent to

$$\delta_c(p) \geq \frac{p - p_c}{p}, \ p \geq p_c. \tag{1.4}$$

(In fact, it is conjectured in that paper that for self-destructive percolation on these lattices $\delta_c(p)$ does not go to 0 as $p \downarrow p_c$, but a proof or disproof of that conjecture seems out of reach at the moment).

Using totaly different arguments it has been proved that (1.4) also holds for site percolation on the binary tree (see Theorem 2.6.1 in [3] or Theorem 5.1 in [1]). This raises the question how general (1.4), or, at least, the following weaker property, are:

$$\exists \hat{p} > p_c \ \exists C > 0 \text{ s.t. } \ \forall p \in [p_c, \hat{p}] \ \delta_c(p) \geq C(p - p_c). \tag{1.5}$$

As mentioned in Remark (ii) just after the above mentioned Proposition 3.1 in [1], the argument of this Proposition does not work for bond percolation on the square lattice. However, in Section 2 we will show that the statement of that Proposition is true for that model. We do this by refining the argument in [1].

On the other hand, it is easy to see that (1.4) can not hold for the bond model on the triangular lattice, the site model on the matching (or star) lattice of the square lattice, and more generally any percolation model with p_c smaller than 1/2: For such models $(p - p_c)/p$ is clearly larger than 1/2

(and hence larger than p_c) for p sufficiently close to 1. For such p (1.4) would contradict (1.2).

Nevertheless, still further refinement of the arguments shows that for a large class of 2D percolation models, including the above mentioned bond model on the triangular lattice and the site model on the star lattice of the square lattice, (1.5) does hold. This is done in Section 3.

This paper focuses mainly on transitive graphs. However, related to Remark 6 at the end of Section 3, we make the following comment on non-transitive graphs. On such graphs $P_p(v \leftrightarrow \infty)$ may depend on the vertex v, and for this reason the notation $\theta(p; v)$ is used. However (as is well-known), by positive association (FKG) $\theta(p; w) \geq P_p(v \leftrightarrow w)\theta(p; v)$ for all vertices v, w. Therefore $\theta(p; v)$ is positive if and only if $\theta(p; w)$ is positive, and hence p_c does not depend on v. For similar reasons (see [1] for positive association for the self-destructive percolation model) $\delta_c(p)$ does not depend on v either.

2 Bond percolation on the Square Lattice

Let $\mathbb{L}^2 = (\mathbb{Z}^2, \mathbb{E}(\mathbb{L}^2))$ be the square lattice, where

$$\mathbb{E}(\mathbb{L}^2) = \{\langle x, y\rangle; \|x - y\|_1 = 1\}$$

and let $G_{cb} = (\mathbb{Z}^2, \mathbb{E}_{cb})$ be the chess-board lattice, where

$$\mathbb{E}_{cb} = \{\langle x, y\rangle; \|x - y\|_1 = 1\}$$

$$\cup \ \{\langle (x_1, x_2), (y_1, y_2)\rangle; (y_1, y_2) = (1, 1) + (x_1, x_2) \text{ and } x_1 + x_2 \text{ is even } \}$$

$$\cup \ \{\langle (x_1, x_2), (y_1, y_2)\rangle; (y_1, y_2) = (1, -1) + (x_1, x_2) \text{ and } x_1 + x_2 \text{ is odd } \}$$

It is well-known that bond percolation on the square lattice is equivalent to site percolation on its covering graph (see section 2.5 in [5] or section 1.6 in [4]), the chess-board lattice.

The next result is an extension of Proposition 3.1 in [1] for bond percolation on the square lattice. For the proof of that proposition it was essential that the lattice under consideration is a subgraph of its matching lattice. (For the notion 'matching lattice', see e.g. Section 3.1 in [4]. In the particular case of the square lattice, the matching lattice is obtained from the square lattice by adding, in each face, the two diagonals as extra edges; the matching lattice of the triangular lattice is the tringular lattice itself: it is

self-matching). The chess-board lattice does not have this property. This is why the proof of [1] does not work for the site model on that lattice (and hence for bond percolation on the square lattice). However, the chess-board lattice is a translation ('along an edge') of its matching lattice, and we will exploit this property to modify the proof of the above mentioned Proposition 3.1 in [1].

Theorem 1. *For the self-destructive site percolation model on the chess-board lattice (or, equivalently, bond percolation on the square lattice), it holds that if $p(1-\delta) > p_c$ then $\theta(p, \delta) = 0$. Hence, $\delta_c(p) \geq \frac{p-p_c}{p}$.*

Proof. Given $v \in \mathbb{Z}^2$, we will use the notation $\tilde{v} := v + (1, 0)$. Recall the sequences of 0-1 valued random variables, X_v, Y_v, Z_v $v \in \mathbb{Z}^2$, introduced in section 1. We color each vertex $v \in \mathbb{Z}^2$ red if $X_v = 1$ and $Y_{\tilde{v}} = 0$. Then, each vertex v will be red with probability $p(1-\delta)$, independently of the other vertices. Since, $p(1-\delta) > p_c$ it follows from ordinary site percolation (see section 11.8 in [4]), that, a.s., there is an infinite red cluster, and this cluster contains a circuit around the origin.

Let γ be such a red circuit. Define

$$\tilde{\gamma} := \gamma + (1, 0).$$

Note that $\tilde{\gamma}$ is a circuit in the matching graph. Let $\tilde{v} \in \tilde{\gamma}$. By construction, $Y_{\tilde{v}} = 0$. Moreover, since $\tilde{v}$ is a neighbor of the infinite X-occupied cluster, $X^*_{\tilde{v}} = 0$. Hence $Z_{\tilde{v}} = 0$. Summarizing we have that, almost surely, there is a Z-vacant circuit in the matching graph which surrounds or contains the origin. Hence $\theta(p, \delta) = 0$. □

3 Other $2D$ lattices

Note that in the proof of Theorem 1 (as in that of Proposition 3.1 of [1]) the definition of *red* vertices (or edges) was done in such a way that each vertex (edge) is red independently of the other vertices (edges). In the current Section the color of a vertex will involve the Y values of all its neighbours. This strategy, which will be used to show that (1.5) is true for a large class of $2D$ lattices, leads to dependencies which somewhat complicate the analysis.

The main result of this section, Theorem 2 below, is stated for three well-known lattices, but, as we shall point out in Remark 6, the result (with practically the same proof) holds for a large class of $2D$ lattices.

Theorem 2. *For self-destructive site percolation on the matching lattice of the square or honeycomb lattice, or self-destructive bond percolation on the triangular lattice, the following holds:*

There are $\hat{p} > p_c$ and $C > 0$ such that

$$\forall p \in [p_c, \hat{p}]\, \delta_c(p) \geq C(p - p_c).$$

Proof. From now on $G = (\mathbb{V}, \mathbb{E})$ denotes the matching lattice of the square or honeycomb lattice, or the covering lattice of the triangular lattice, and p_c the critical probability for site percolation on G. (See, however, Remark 6 below).

Recall that for the proofs in Section 2 we introduced a certain colouring of the vertices, and that the colours were i.i.d. so that we could compare the result of the coulouring with ordinary percolation. In the current situation we will again define a colouring, but now the colours are not independent. Nevertheless it turns out that we again obtain a suitable comparison with ordinary percolation. First some notation.

For each vertex $v \in \mathbb{V}$, let $D_v := \{u \in \mathbb{V}; \langle v, u \rangle \in \mathbb{E}\}$. Let $d := |D_v|$ (for example, $d = 8$ for the matching lattice of the square lattice).

Recall the sequences $X_v, Y_v,\ v \in \mathbb{V}$ defined in Section 1. Now define the sequence of 0-1 random variables $R_v,\ v \in \mathbb{V}$ as

$$R_v = \begin{cases} 1, \text{ if } X_v = 1 \text{ and } Y_u = 0,\ \forall u \in D_v \\ 0, \text{ otherwise} \end{cases} \tag{3.6}$$

If $R_v = 1$ we say that v is $R-$occupied (or, simply, that v is red).

Before we proceed with the proof of the theorem, we first state Observation 3 and state and prove Lemma 4 below, which will be used later.

Observation 3. *Let γ be a circuit in G. Then every path that starts in the interior of γ and ends in the exterior of γ contains a vertex which has a neighbour on γ.*

Lemma 4. *Let $\epsilon > 0$. There is a constant c_ϵ such that for the self-destructive site percolation model on G with parameters $0 < \delta \leq p_c$ and $p \in (p_c, 1 - \epsilon)$*

the following holds: For every $v \in \mathbb{V}$, every finite subset of vertices $F \subset \mathbb{V}$ and every colouring $(r_u,\, u \in F)$ of F,

$$\frac{P(Y_v = 1; R_u = r_u,\, u \in F)}{P(Y_v = 0; R_u = r_u,\, u \in F)} \leq c_\epsilon \delta.$$

Hence, $P(Y_v = 1 | R_u = r_u,\, u \in F) \leq c_\epsilon \delta$.

Proof of Lemma 4: If $D_v \cap F = \emptyset$, the random variable Y_v is independent of the sequence R_u, $u \in F$, and hence

$$\frac{P(Y_v = 1;\, R_u = r_u, u \in F)}{P(Y_v = 0;\, R_u = r_u, u \in F)} = \frac{\delta}{1-\delta}.$$

For the case $F' := D_v \cap F \neq \emptyset$, we consider two subcases: If there is at least one $u \in F'$ with $r_u = 1$ then obviously

$$P(Y_v = 1, R_u = r_u,\; \forall u \in F) = 0,$$

and it is easy to see that $P(Y_v = 0;\, R_u = r_u, u \in F) > 0$.

If $r_u = 0,\; \forall u \in F'$ we have that

$$\begin{aligned} P(Y_v = 1;\, R_u = r_u, u \in F) &= P(Y_v = 1) P(R_u = r_u, u \in F \,|\, Y_v = 1) \\ &\leq\; \delta P(R_u = r_u, u \in F \setminus F' \,|\, Y_v = 1) \\ &=\; \delta P(R_u = r_u, u \in F \setminus F'), \end{aligned}$$

and

$$\begin{aligned} P(Y_v = 0;\, R_u = r_u, u \in F) &= P(Y_v = 0) P(R_u = r_u, u \in F \,|\, Y_v = 0) \\ &\geq\; (1-\delta) P(R_u = r_u, u \in F;\, X_z = 0, z \in D_v \,|\, Y_v = 0) \\ &=\; (1-\delta)(1-p)^d P(R_u = r_u, u \in F \,|\, Y_v = 0;\, X_z = 0, z \in D_v) \\ &=\; (1-\delta)(1-p)^d P(R_u = r_u, u \in F \setminus F'). \end{aligned}$$

Combining these two inequalities we have that

$$\frac{P(Y_v = 1;\, R_u = r_u, u \in F)}{P(Y_v = 0;\, R_u = r_u, u \in F)} \leq \frac{\delta}{(1-\delta)(1-p)^d}.$$

So the claim of the lemma holds, with the constant $c_\epsilon = \frac{1}{(1-p_c)\,\epsilon^d}$. *This completes the proof of Lemma* 4.

Now we continue the proof of Theorem 2. Let $\epsilon > 0$. Suppose $p \in (p_c, 1-\epsilon)$, and let δ be such that $p(1 - dc_\epsilon\delta) > p_c$, where c_ϵ is the constant given in Lemma 4.

Let $v \in \mathbb{V}$, F a finite set of vertices not containing v and $r_u \in \{0, 1\}$, $\forall u \in F$. We have

$$\begin{aligned} P(R_v = 1 \,|\, R_u = r_u, u \in F) &= P(X_v = 1;\, Y_z = 0, z \in D_v \,|\, R_u = r_u, u \in F) \\ &= P(X_v = 1) P(Y_z = 0,\, z \in D_v \,|\, R_u = r_u, u \in F) \\ &\geq p(1 - dc_\epsilon\delta), \end{aligned}$$

where in the inequality we used Lemma 4 (and in the equality the fact that X_v is independent of the collection of random variables $\{R_u, u \in F; Y_z, z \in D_v\}$). As $p(1 - dc_\epsilon\delta) > p_c$, the process $(R_v, v \in \mathbb{V})$ dominates an i.i.d. process with parameter larger than p_c. Comparison with ordinary percolation shows that a.s. there is an infinite R-occupied cluster which contains a circuit around the origin. Let γ be such a circuit. Observe that, by the definition of the colourings, γ belongs to an infinite X-occupied cluster. Define

$$\Gamma := \cup_{v \in \gamma} D_v.$$

For each $w \in \Gamma$ we have either $X_w = 1$, in which case (by the above observation) w belongs to an infinite $X-$open cluster, or we have $X_w = 0$. In both cases $X^*_w = 0$. Since also $Y_w = 0$ for all $w \in \Gamma$, we have $Z_w = 0$ for all $w \in \Gamma$. By Observation 3 we now conclude that there is no infinite $Z-$open path starting in O. So we have proved that for all $\epsilon > 0$ and all $p < 1 - \epsilon$, it holds that $\theta(p, \delta) = 0$ if $p(1 - d\, c_\epsilon\, \delta) > p_c$; that is,

$$\delta_c(p) \geq \frac{p - p_c}{p\, d\, c_\epsilon},\ \ p < 1 - \epsilon. \tag{3.7}$$

This completes the proof of Theorem 2. $\square$

Remark 5. *Note that in fact we have proved something stronger than the claim in the theorem, namely that for every $\epsilon > 0$ there is a $c_\epsilon > 0$ such that* (3.7) *holds for all $p < 1 - \epsilon$.*

This can be extended to the result that there is a $C > 0$ such that $\delta_c(p) \geq C(p - p_c$ for all $p > p_c$. Since the c_ϵ in (3.7) *goes to ∞ as ϵ goes to* 0*, this result does not follow immediately. However, to get the extension, it suffices to show that there is an $\epsilon > 0$ such that $\delta_c(p)$ is bounded away from* 0 *for $p > 1 - \epsilon$. Or, equivalently, that there are $\hat{p} < 1$ and $\delta > 0$ such that*

$$\theta(p,\delta) = 0 \textit{ for all } p > \hat{p}. \tag{3.8}$$

This can be proved by (e.g.) arguments very similar to those used in the proof that (ii) implies (i) in Theorem 5.1 in [2]. Artem Sapozhnikov (private communication) has pointed out to us that further refinements of such arguments show that, on the d-dimensional cubic lattice, $\delta_c(p) \to p_c$ *as* $p \to 1$.

Remark 6. *Theorem 2 holds for a large class of 2D lattices. Essentially we only used that for supercritical percolation the infinite cluster a.s. contains a circuit around O, which satisfies Observation 3. This property holds for site percolation on the lattices belonging to the family in Theorem 12.1 in Kesten's book [5]. Informally speaking, this family consists of lattices which belong to a pair of matching lattices with certain periodicity and reflection symmetry properties (but which are not necessarily transitive).*

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